

Stress Distribution in a Semi-Infinite Body Symmetrically Loaded Over a Circular Area

H. McGinness
DSN Engineering Section

Algorithms are developed for computing stresses in a semi-infinite body when loaded by a uniform pressure acting over a circular area.

I. Introduction

The stress distribution through a semi-infinite body of isotropic material has been obtained in various ways. For the particular case where the loading on the plane surface is one of uniform pressure acting on a circular area, the stress components can be calculated by numerically evaluating the integral expressions presented herein. These integrals can be evaluated by desk calculator programs, thus making it easy to determine any stress component in a semi-infinite body having a known constant Poisson's ratio. The solution of this problem has a direct application to circular columns resting on large footings and can be helpful in estimating stresses in foundations supporting certain vehicle rails.

II. Derivation of the Algorithms

If a concentrated force P is applied perpendicularly to the plane surface of a semi-infinite body, the stress components per Ref. 1 are:

$$\sigma_r = \frac{P}{2\pi} \left\{ (1 - 2\nu) \left[\frac{1}{r^2} - \frac{Z}{r^2} (r^2 + Z^2)^{-1/2} \right] - 3r^2 Z (r^2 + Z^2)^{-5/2} \right\} \quad (1)$$

$$\sigma_\theta = \frac{P}{2\pi} (1 - 2\nu) \left\{ -\frac{1}{r^2} + \frac{Z}{r^2} (r^2 + Z^2)^{-1/2} + Z (r^2 + Z^2)^{-3/2} \right\} \quad (2)$$

$$\sigma_z = -\frac{3P}{2\pi} Z^3 (r^2 + Z^2)^{-5/2} \quad (3)$$

$$\tau_{rz} = -\frac{3P}{2\pi} r Z^2 (r^2 + Z^2)^{-5/2} \quad (4)$$

where the coordinates are defined as follows: Assume the plane of the semi-infinite body to be horizontal and on the upper side of the body. The coordinate r is the horizontal distance from the point of concern, point 0, to the point directly beneath the force P . The positive coordinate Z is the distance that point 0 is below the plane. The coordinate θ is mutually perpendicular to r and Z . The symbols σ and τ represent the normal and shear stresses, respectively, and ν is Poisson's ratio.

The solution for the case of a uniform pressure acting over a circular area of radius a is obtained by replacing the concentrated load P by $p dA$ where p is the uniform pressure and dA is a differential area, and appropriately summing over the circular area.

Let the point 0 be on a vertical plane which passes through the center of the circularly loaded area, as shown in Fig. 1. The differential area is $r d\theta dr$. It is desired to obtain the stress components along axes R , T , and Z , namely, σ_R , σ_T , σ_Z and the shear stress τ_{RZ} . By symmetry the other shear stresses will be zero.

Thus from these four components, principal stresses can be calculated. By the usual method of resolving two-dimensional stress components the following obtain:

$$\sigma_R = \sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta \quad (5)$$

$$\sigma_T = \sigma_r \sin^2 \theta + \sigma_\theta \cos^2 \theta \quad (6)$$

$$\tau_{RZ} = \tau_{rz} \cos \theta \quad (7)$$

The stress components σ_R , σ_T , σ_Z , and τ_{RZ} caused by the pressure loading over a differential area are formed by substituting $prdrd\theta$ for P in Eqs. (1), (2), (3), and (4) and substituting the results into Eqs. (5), (6), and (7). The following double integrals represent the desired stress components in terms of the coordinates r , θ , and Z :

$$\begin{aligned} \frac{\sigma_R}{p} = & \frac{1-2\nu}{2\pi} \int_{\phi_1}^{\phi_2} \int_L^U \left\{ \cos^2 \theta \left[\frac{1}{r} - \frac{Z}{r(r^2 + Z^2)^{1/2}} \right. \right. \\ & - \frac{3r^3 Z}{(1-\nu)(r^2 + Z^2)^{5/2}} \left. \right] + \sin^2 \theta \left[-\frac{1}{r} + \frac{Z}{r(r^2 + Z^2)^{1/2}} \right. \\ & \left. \left. + \frac{rZ}{(r^2 + Z^2)^{3/2}} \right] \right\} dr d\theta \quad (8) \end{aligned}$$

$$\begin{aligned} \frac{\sigma_T}{p} = & \frac{1-2\nu}{2\pi} \int_{\phi_1}^{\phi_2} \int_L^U \left\{ \sin^2 \theta \left[\frac{1}{r} - \frac{Z}{r(r^2 + Z^2)^{1/2}} \right. \right. \\ & - \frac{3r^3 Z}{(1-\nu)(r^2 + Z^2)^{5/2}} \left. \right] + \cos^2 \theta \left[-\frac{1}{r} + \frac{Z}{r(r^2 + Z^2)^{1/2}} \right. \\ & \left. \left. + \frac{rZ}{(r^2 + Z^2)^{3/2}} \right] \right\} dr d\theta \quad (9) \end{aligned}$$

$$\frac{\sigma_Z}{p} = -\frac{3}{2\pi} \int_{\phi_1}^{\phi_2} \int_L^U \frac{rZ^3}{(r^2 + Z^2)^{5/2}} dr d\theta \quad (10)$$

$$\frac{\tau_{RZ}}{p} = -\frac{3}{2\pi} \int_{\phi_1}^{\phi_2} \int_L^U (\cos \theta) \frac{r^2 Z^2}{(r^2 + Z^2)^{5/2}} dr d\theta \quad (11)$$

Equations (8), (9), (10), and (11) are readily integrated with respect to r to yield:

$$\begin{aligned} \frac{\sigma_R}{p} = & \frac{1-2\nu}{2\pi} \int_{\phi_1}^{\phi_2} \left\{ \cos 2\theta \left[\ln(Z + \sqrt{r^2 + Z^2}) \right]_L^U \right. \\ & + \frac{\cos^2 \theta}{(1-2\nu)} \left[\frac{3Z(r^2 + \frac{2}{3}Z^2)}{(r^2 + Z^2)^{3/2}} \right]_L^U \\ & \left. - \sin^2 \theta \left[\frac{Z}{(r^2 + Z^2)^{1/2}} \right]_L^U \right\} d\theta \quad (12) \end{aligned}$$

$$\begin{aligned} \frac{\sigma_T}{p} = & \frac{1-2\nu}{2\pi} \int_{\phi_1}^{\phi_2} \left\{ -\cos 2\theta \left[\ln(Z + \sqrt{r^2 + Z^2}) \right]_L^U \right. \\ & + \frac{\sin^2 \theta}{(1-2\nu)} \left[\frac{3Z(r^2 + \frac{2}{3}Z^2)}{(r^2 + Z^2)^{3/2}} \right]_L^U \\ & \left. - \cos^2 \theta \left[\frac{Z}{(r^2 + Z^2)^{1/2}} \right]_L^U \right\} d\theta \quad (13) \end{aligned}$$

$$\frac{\sigma_Z}{p} = \frac{1}{2\pi} \int_{\phi_1}^{\phi_2} \left[\frac{Z^3}{(r^2 + Z^2)^{3/2}} \right]_L^U d\theta \quad (14)$$

$$\frac{\tau_{RZ}}{p} = -\frac{1}{2\pi} \int_{\phi_1}^{\phi_2} \cos \theta \left[\frac{r^3}{(r^2 + Z^2)^{3/2}} \right]_L^U d\theta \quad (15)$$

A distinction must be made between Case 1, where point 0 lies under or on the loaded area, as shown in Fig. 1, and Case 2, where point 0 lies outside the loaded area, as shown in Fig. 2.

For Case 1 the point is located distance βa from the boundary of the circle and a distance γa below the surface. The distance from point 0 to point U is OU , where

$$OU = a \left[(1 - \beta) + \sqrt{(1 + \beta)^2 + \frac{\beta(2 - \beta)}{\cos^2 \theta}} \right] \cos \theta \quad (16)$$

The distance from point 0 to point L is OL , where

$$OL = -a \left[(1 - \beta) - \sqrt{(1 + \beta)^2 + \frac{\beta(2 - \beta)}{\cos^2 \theta}} \right] \cos \theta \quad (17)$$

and the positive roots of the radicals are to be used.

In order for the integration to cover the entire circular area for Case 1, the bracketed terms of Eqs. (12), (13), (14), and (15) must be evaluated for two sets of limits and summed, namely between OU and zero and between OL and zero. The limits of θ are $\pi/2$ and zero provided the integrals are multiplied by 2.

For Case 2 the point 0 is located a distance αa from the boundary of the circle and a distance γa below the surface. The upper and lower limits of the bracketed terms of Eqs. (12), (13), (14), and (15) are respectively:

$$u = a \left[(1 + \alpha) + \sqrt{(1 + \alpha)^2 - \frac{\alpha(\alpha + 2)}{\cos^2 \theta}} \right] \cos \theta \quad (18)$$

$$l = a \left[(1 + \alpha) - \sqrt{(1 + \alpha)^2 - \frac{\alpha(\alpha + 2)}{\cos^2 \theta}} \right] \cos \theta \quad (19)$$

The limits of θ are $\arctan \sqrt{1/\alpha(\alpha + 2)}$ and zero provided the integrals are multiplied by 2.

The results are as follows:

For Case 1

$$\frac{\sigma_{1R}}{p} = \frac{1 - 2\nu}{\pi} \int_0^{\pi/2} \left\{ \cos 2\theta \left[\ln \frac{(\gamma + \sqrt{U^2 + \gamma^2})(\gamma + \sqrt{L^2 + \gamma^2})}{4\gamma^2} \right] \right.$$

$$+ \frac{\cos^2 \theta}{1 - 2\nu} \left[\frac{3\gamma \left(U^2 + \frac{2}{3}\gamma^2 \right)}{(U^2 + \gamma^2)^{3/2}} + \frac{3\gamma \left(L^2 + \frac{2}{3}\gamma^2 \right)}{(L^2 + \gamma^2)^{3/2}} - 4 \right] \\ \left. - \sin^2 \theta \left[\frac{\gamma}{(U^2 + \gamma^2)^{1/2}} + \frac{\gamma}{(L^2 + \gamma^2)^{1/2}} - 2 \right] \right\} d\theta \quad (20)$$

$$\frac{\sigma_{1T}}{p} = \frac{1 - 2\nu}{\pi} \int_0^{\pi/2} \left\{ -\cos 2\theta \left[\ln \frac{(\gamma + \sqrt{U^2 + \gamma^2})(\gamma + \sqrt{L^2 + \gamma^2})}{4\gamma^2} \right] \right.$$

$$+ \frac{\sin^2 \theta}{1 - 2\nu} \left[\frac{3\gamma \left(U^2 + \frac{2}{3}\gamma^2 \right)}{(U^2 + \gamma^2)^{3/2}} + \frac{3\gamma \left(L^2 + \frac{2}{3}\gamma^2 \right)}{(L^2 + \gamma^2)^{3/2}} - 4 \right] \\ \left. - \cos^2 \theta \left[\frac{\gamma}{(U^2 + \gamma^2)^{1/2}} + \frac{\gamma}{(L^2 + \gamma^2)^{1/2}} - 2 \right] \right\} d\theta \quad (21)$$

$$\frac{\sigma_{1z}}{p} = \frac{1}{\pi} \int_0^{\pi/2} \left\{ \gamma^3 \left[\frac{1}{(U^2 + \gamma^2)^{3/2}} + \frac{1}{(L^2 + \gamma^2)^{3/2}} \right] - 2 \right\} d\theta \quad (22)$$

$$\frac{\tau_{1RZ}}{p} = -\frac{1}{\pi} \int_0^{\pi/2} (\cos \theta) \left[\frac{U^3}{(U^2 + \gamma^2)^{3/2}} - \frac{L^3}{(L^2 + \gamma^2)^{3/2}} \right] d\theta \quad (23)$$

where

$$U = \left[(1 - \beta) + \sqrt{(1 + \beta)^2 + \frac{\beta(2 - \beta)}{\cos^2 \theta}} \right] \cos \theta \quad (24)$$

$$L = - \left[(1 - \beta) - \sqrt{(1 + \beta)^2 + \frac{\beta(2 - \beta)}{\cos^2 \theta}} \right] \cos \theta \quad (25)$$

For Case 2

$$\frac{\sigma_{2R}}{p} = \frac{1 - 2\nu}{\pi} \int_0^{\phi} \left\{ \cos 2\theta \left[\ln \frac{(\gamma + \sqrt{U^2 + \gamma^2})}{(\gamma + \sqrt{l^2 + \gamma^2})} \right] \right. \\ + \frac{(\cos^2 \theta)(3\gamma)}{1 - 2\nu} \left[\frac{U^2 + \frac{2}{3}\gamma^2}{(U^2 + \gamma^2)^{3/2}} - \frac{l^2 + \frac{2}{3}\gamma^2}{(l^2 + \gamma^2)^{3/2}} \right] \\ \left. - (\sin^2 \theta)(\gamma) \left[\frac{1}{(U^2 + \gamma^2)^{1/2}} - \frac{1}{(l^2 + \gamma^2)^{1/2}} \right] \right\} d\theta \quad (26)$$

$$\begin{aligned} \frac{\sigma_{2T}}{p} = & \frac{1-2\nu}{\pi} \int_0^\phi \left\{ -\cos 2\theta \left[\ln \frac{(\gamma + \sqrt{u^2 + \gamma^2})}{(\gamma + \sqrt{l^2 + \gamma^2})} \right] \right. \\ & + \frac{(\sin^2 \theta)(3\gamma)}{1-2\nu} \left[\frac{u^2 + \frac{2}{3}\gamma^2}{(u^2 + \gamma^2)^{3/2}} - \frac{l^2 + \frac{2}{3}\gamma^2}{(l^2 + \gamma^2)^{3/2}} \right] \\ & \left. - (\cos^2 \theta)(\gamma) \left[\frac{1}{(u^2 + \gamma^2)^{1/2}} - \frac{1}{(l^2 + \gamma^2)^{1/2}} \right] \right\} d\theta \quad (27) \end{aligned}$$

$$\frac{\sigma_{2Z}}{p} = \frac{\gamma^3}{\pi} \int_0^\phi \left[\frac{1}{(u^2 + \gamma^2)^{3/2}} - \frac{1}{(l^2 + \gamma^2)^{3/2}} \right] d\theta \quad (28)$$

$$\frac{\tau_{2RZ}}{p} = -\frac{1}{\pi} \int_0^\phi (\cos \theta) \left[\frac{u^3}{(u^2 + \gamma^2)^{3/2}} - \frac{l^3}{(l^2 + \gamma^2)^{3/2}} \right] d\theta \quad (29)$$

where

$$u = \left[(1 + \alpha) + \sqrt{(1 + \alpha)^2 - \frac{\alpha(\alpha + 2)}{\cos^2 \theta}} \right] \cos \theta \quad (30)$$

$$l = \left[(1 + \alpha) - \sqrt{(1 + \alpha)^2 - \frac{\alpha(\alpha + 2)}{\cos^2 \theta}} \right] \cos \theta \quad (31)$$

$$\phi = \arctan \sqrt{\frac{1}{\alpha(\alpha + 2)}} \quad (32)$$

The integrals of Eqs. (20), (21), (22), (23), (26), (27), (28), and (29) can easily be integrated numerically on programmable desk calculators. Angular increments of 1/9 the angular range will produce results sufficiently accurate for most engineering applications. These programs are available on four cards for HP97 calculators.

Principal stresses can be calculated per Ref. 2 as follows:

$$\sigma_{P1} = \frac{1}{2} \left[\sigma_R + \sigma_Z + \sqrt{(\sigma_R - \sigma_Z)^2 + 4\tau_{RZ}^2} \right] \quad (33)$$

$$\sigma_{P2} = \frac{1}{2} \left[\sigma_R + \sigma_Z - \sqrt{(\sigma_R - \sigma_Z)^2 + 4\tau_{RZ}^2} \right] \quad (34)$$

$$\sigma_{P3} = \sigma_T \quad (35)$$

III. Discussion of Results

On the plane surface of the semi-infinite body the stresses σ_R , σ_T , and σ_Z are discontinuous at the boundary of the loaded circular area. Even at finite values of γ , that is, for points beneath the plane surface, the algorithms fail when the points are directly beneath the boundary. However, in Ref. 2 it is demonstrated that there are no infinite stress values obtained at the boundary and that all stresses are continuous at the boundary except those at the surface. Therefore, the values at the boundary may be approximated by considering very small values of the parameters α or β . For these reasons it is convenient to plot the calculated stresses on a semi-logarithmic chart with the abscissas α and β extending in opposite directions from a common small value. In Fig. 3 this has been done for a Poisson's ratio of 0.15, starting with α and β values of 0.001. Figure 3 gives the stresses at various distances below the plane surface, that is, at γ values of 0, 0.001, 0.01, 0.10, and 1.00. The ranges are sufficient to show the value and location of the maximum tensile stresses.

The curves of Fig. 3 pertain to a Poisson's ratio of 0.15 because this is a typical value for a Portland cement grout. Such grout has a tensile strength far less than its compressive strength. The curves can be useful in determining what tensile strength is necessary to ensure that surface cracks are not likely to form, and how far below the surface the high tensile stress region extends.

The curves of Fig. 4 pertain to a Poisson's ratio of 0.30, representative of many metals. Only the stress components σ_R and σ_T are shown, since σ_Z and τ_{RZ} are independent of Poisson's ratio and may be taken from Fig. 3. By comparing corresponding curves of Figs. 3 and 4 it is seen that the effect of Poisson's ratio on σ_R and σ_T is large.

Results obtained with the above algorithms, by dividing the angular range into 9 parts, agree with the tabulated values of Ref. 2.

By superposition the stresses can be calculated for any circularly symmetric loading. For example, the real loading could be approximated by a number of uniform loads of different radii, and the effects of each appropriately summed by the above algorithms.

References

1. Timoshenko, S., *Theory of Elasticity*, McGraw Hill, 1934, First edition, p. 331, Eqs. 196.
2. Love, A. E. H., "The Stress Produced in a Semi-Infinite Solid by Pressure on Part of the Boundary," *Phil. Trans. Roy. Soc., London, Series A*, Vol. 228, 1929.

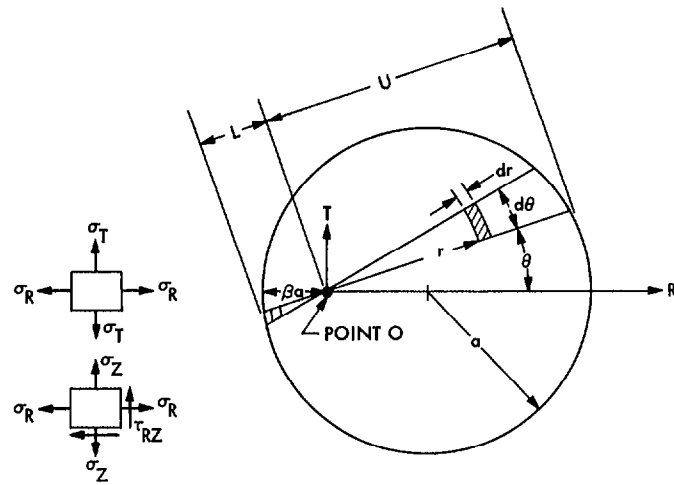


Fig. 1. Integration limits when point O lies below loaded circular area

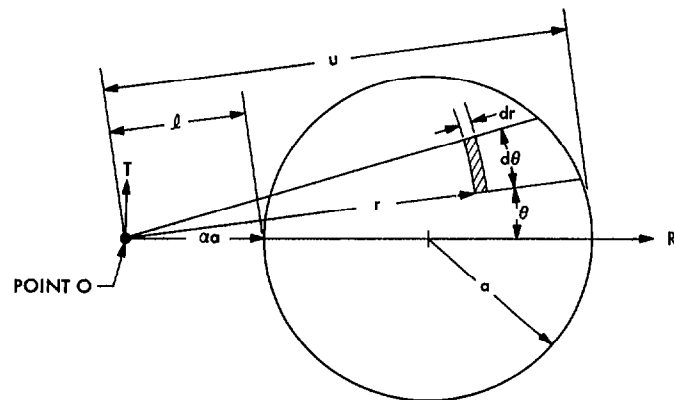


Fig. 2. Integration limits when point O lies outside loaded circular area

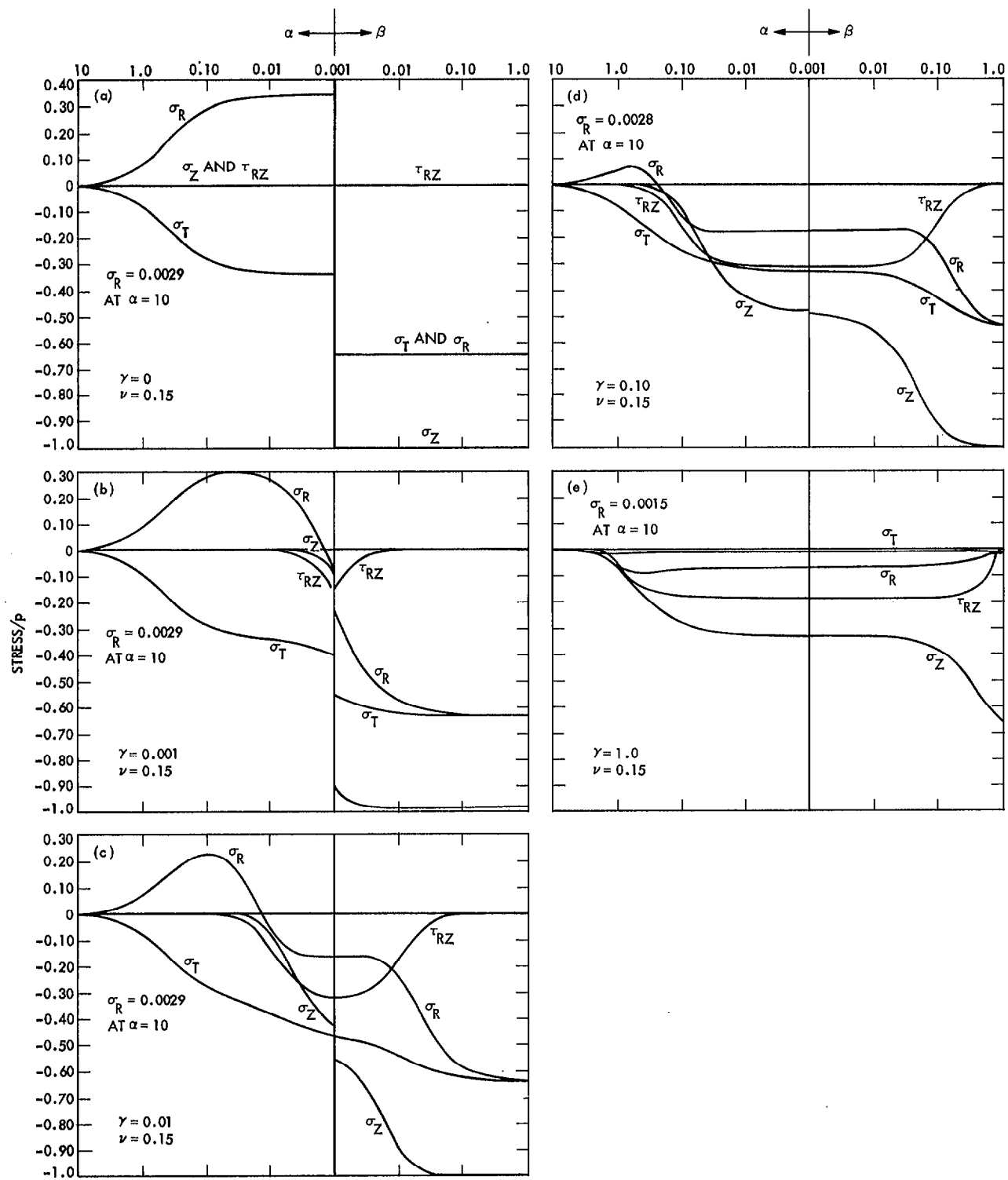


Fig. 3. Stress ratios vs α & β , $\nu = 0.15$

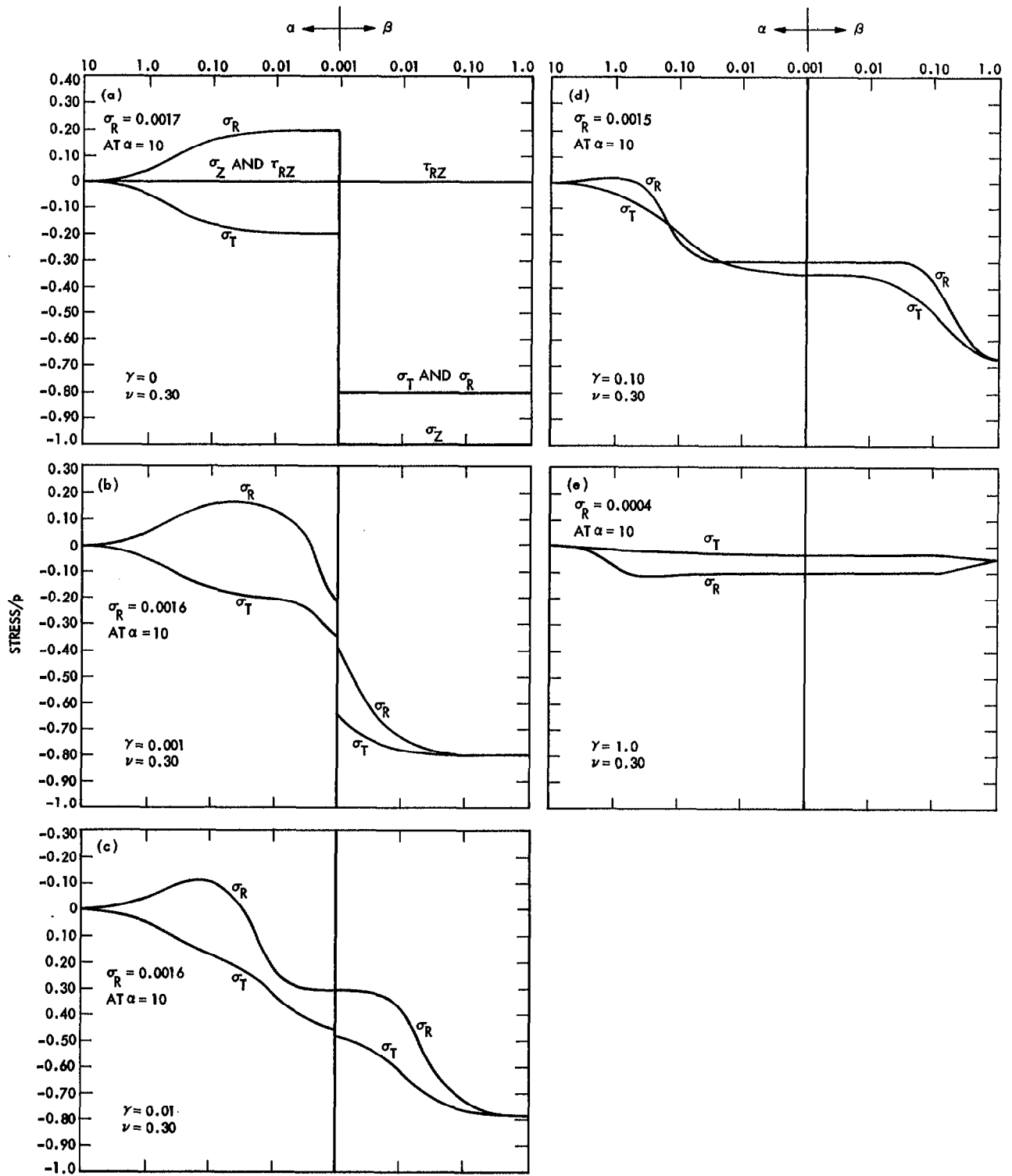


Fig. 4. Stress ratios vs α & β , $\nu = 0.30$